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Constant-Stress Partially Accelerated Life Tests of Kumaraswamy Distribution with Generalized Type-I Hybrid Censoring

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Abstract

This paper discusses various methods for estimating the parameters of the Kumaraswamy (Kw) distribution and the acceleration factor in constant stress partially accelerated life test (CSPALTs) using Type-I generalized hybrid censored data. The constant stress partially accelerated life test CSPALTs using censored data that is Type-I generalized hybrid is explained in detail. In classical inference, maximum likelihood (ML) estimates are derived for both the model parameters of the Kumaraswamy (Kw) and the acceleration factor. The Fisher Information Matrix (FIM) is utilized to derive the approximate confidence intervals (ACIs) for all model parameters and the acceleration factor. The model parameters and the acceleration factor are estimated using bootstrap approach. Markov chain Monte Carlo (MCMC) method is suggested for Bayesian inference in order to optimize the related credible intervals (CRIs) and the Bayes estimators. In addition to, symmetric loss functions as well as asymmetric ones are considered. A collection of simulated data is analyzed to demonstrate the viability and appropriateness of the proposed methods. Furthermore, a Monte Carlo simulation is conducted to assess the effectiveness of the proposed methods.

Keywords: Kumaraswamy distribution; constant-stress partially accelerated life tests; generalized Type-I hybrid; ML estimation; bootstrap; MCMC.

1 Introduction

Since many electro-mechanical materials and objects have reached contemporary reliability standards, testing their failure times under typical operating settings is not practical because the products typically have long lives, and extensive applied tests are typically prohibitively expensive. Because of this, it is preferable for the manufacturing industry to conduct accelerated life tests (ALTs) to gather sufficient failure data, in a short period, to draw conclusions about the failure's relationship to external stress variables. The purpose of ALTs is to cause early failures by testing the test items only under accelerated conditions, i.e., elevated stress levels compared to normal. The life span distribution at normal use conditions is estimated by extrapolating data obtained under such accelerated conditions through a physically acceptable statistical model. The partially accelerated life test (PALT) is a specific type of Accelerated Life Testing (ALT) scheme. Pascual [19], studies the planning of ALT in the presence of competing risks.

In PALTs, stress loading can be implemented in various forms, with the most commonly used approaches being constant-stress, step-stress, and progressive-stress methods. Many writers have written on step-stress ALT, including Fan et al. [11] who discussed a $k-$stage step-stress accelerated life-test with M-stress variables under progressive Type-I. In constant stress partially accelerated life test (CSPALTs) are conducted under both normal and accelerated stress conditions to obtain additional failure data within a limited testing period, while preventing the exposure of all units to excessive stress levels. Abdel-Hamid [2] who presented Burr Type-XII distribution based on constant-partially-ALTs under progressive Type-II censoring. Abushal and Soliman [4] obtained estimating the pareto parameters for constant-partially-ALTs under progressive censoring. The constant-partially-ALTs of weibull inverted exponential distribution under progressive first-failure censoring was investigated by Fathi et al. [12]. Abd El-Azeem et al. [1], obtained the Bayes estimate of constant-stress acceleration based on competing risks with progressive Type-II censoring.

In recent decades, a variety of censorship techniques have been employed. The most popular censorship schemes are still Type-I and Type-II. In Type-I censoring, the amount of failures is random and the test ends at a predetermined time T. When using Type-II censoring, the testing period is arbitrary. and terminates following a set certain amount of breakdowns. Within such 2 varieties, the researcher is unsure on if to conclude the experiment (as in Type-II) perhaps the researcher might achieve the required amount of failures ahead of schedule T (as in Type-I). To overcome such constraints of Type-I and Type-II, Epstein [10] created a hybrid censoring scheme (HCS), which primarily consists of Type-I HCS and Type-II HCS. The test in Type-I HCS is ended whenever, $T^* = \min\{X_{r,n}T\}$, since $r \in \{1, 2, \ldots, n\}$ in addition to $T \in (0, \infty)$ there set up right at the first moment for the examination. With respect to Type-II HCS, the test ends randomly, and consider it $T^* = \max\{X_{r,n}T\}$. Such methods also possess restrictions, such as limited quantity for losses or an inability to determine the maximum time allotted for the test.

Therefore, Chandrasekar et al. [6] considered a hybrid censoring scheme and obtained the exact lower confidence. Classical and Bayesian estimation under Type-I progressive hybrid censoring was introduced by Kishan and Sangal [16]. To address the shortcomings of HCSs, a novel and effective censoring technique called as generalized hybrid censoring schemes (GHCSs) was presented. Type-I and Type-II HCSs are thought to be extended by these approaches. As a result, it is evident that two categories of censorship systems are established in this manner. Type-I Generalized HCS(Type-I GHCS): allow $k, r \in \{1, 2, \ldots, n\}$ are integer numbers and $k < r < n$, using a time period $T \in (0, \infty)$. This plan concludes the test at $\min\{X_{r,n}, T\}$ when the $k-$th failure occurs prior to T. For $k-$th disaster happened next to T, the test is stopped at $X_{k,n}$. Type-II Generalized HCS(Type-II GHCS): set $r \in \{1, 2, \ldots, n\}$ using a time period $T_1, T_2 \in (0, \infty)$, where

$T_1 < T_2$. If $r-$th failure happens faster than expected T_1, the test is finished at T_1. If the $r-$th failure is acquired between T_1 and T_2, the experiment is completed at $X_{r,n}$. In the end, if the $r-$th failure happens gradually. T_2, the experiment is completed at T_2. Several researchers have examined these processes. Huang and Yang [15], who proposed a new censoring scheme for saving time. Rabie and Li. [20, 21] obtained the E-bayesian estimation based on generalized Type-II hybrid censored data and generalized Type-I hybrid censoring scheme. Inferences of accelerated generalized Type-I hybrid censoring was introduced by Bakoban et al. [5].

Kumaraswamy distribution is from the family of continuous distributions, its used in many natural phenomena, such as hydrological data, economic data (like unemployment statistics), and atmospheric temperatures, scores obtained in a test, there are lower and upper limits on the results. The Kumaraswamy distribution was suggested by Kumaraswamy [17]. Nadarajah [18] showed that the Kumaraswamy distribution is a special case of the generalized beta distribution. Classical and Bayesian estimations on the Kumaraswamy distribution under difference loss functions was introduced by Gholizadeh et al. [13]. Abou-Elheggag et al. [3] discussed the estimation of $R = P(Y < X)$ of Kumaraswamy distribution under $k-$upper record values.

The cumulative distribution function (CDF), probability density function (PDF), reliability function (SF) and hazard rate function (HRF) for $Kw(\alpha, \beta)$ distribution are expressed, correspondingly, as,

$$\begin{aligned}
F_1(x;\alpha,\beta) &= 1-(1-x^{\alpha})^{\beta}\,, && 0<x<1, \quad (\alpha,\beta>0)\,,\\
f_1(x;\alpha,\beta) &= \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\,, && 0<x<1, \quad (\alpha,\beta>0)\,,\\
S_1(x) &= (1-x^{\alpha})^{\beta}\,, && 0<x<1, \quad (\alpha,\beta>0)\,,
\end{aligned} \tag{1}$$

and

$$H_1(x) = \frac{\alpha\beta x^{\alpha-1}}{(1-x^{\alpha})}, \qquad 0<x<1, \quad (\alpha,\beta>0)\,,$$

where $\alpha, \beta > 0$ are shape parameters.

This paper aims to estimate the parameters for kumaraswamy distribution and the acceleration factor under constant stress partially accelerated life test (CSPALTs) model using Type-I Generalized HCS(Type-I GHCS). This paper's remaining sections are arranged as described below: The proposed model is described in detail in Section 2. The MLEs for the appropriate parameters are obtained in Section 3. In Sections 4, the Bayesian estimation is discussed based on the square error loss function (SELF) and linear exponantion (LINEX) loss functions by using the MCMC method. In Section 5, interval estimation is constructed for approximate confidence intervals. Two parametric and estimated bootstrap confidence intervals (CIs) are included in Section 6. Section 7 provides an explanation using a numerical example. Section 8 presents the findings of a simulation study that are used to compare the various methods. Finally, Section 9 presents the conclusions.

2 Overview of the Model and Test Procedures

In constant stress partially accelerated life test ($CSPALTs$), we divide a sample of size into two groups n_1 and $n_2 = (n - n_1)$ which are selected randomly, Every group is only operated under accelerated conditions or normal usage conditions at the same time. used conditions under usual and accelerated, the test is continuing at most until time T. Under Type-I GHCS assume that, $X_{ji}, j = \{1, 2\}; i = \{1, 2, \ldots, n_j\}$ are the products' units that obtained from Kw distribution, such as $X_{1i}; i = 1, \ldots, n_1$ is the lifetime of the unit in normal condition and $X_{2i}; i = 1, \ldots, n_2$ is the lifetime in accelerated condition. Denote $X_{(j1)} \leq X_{(j2)} \leq \cdots \leq X_{(jnj)}$ as the order statistics of X_{ji}, $i = 1, 2, \ldots, nj$, respectively. Fix $l_j, m_j \in \{1, 2, \ldots, n_j\}$ and $T \in (0, \infty)$ such that $1 < l_j < m_j < n_j$. When l_j^{th} failure occurs prior to time T, stopping the test is necessary at $min\{X_{m_j:n_j}, T\}$. When l_j^{th} failure happens after time T, the test should be stopped at $X_{m_j:n_j}$. Under this setting, although a bare minimum of l_j failures is acceptable, the researcher would prefer to observe m_j failures. Realization for the failure time sample can be presented by $X_{(j1)} \leq X_{(j2)} \leq \cdots \leq X_{(jD_j)}$, D_j represents The quantity of failures happened prior to time point T, where

$$D_j = \begin{cases} l_j, & \text{if} \quad X_{l_j} > T, \\ d_j^*, & \text{if} \quad X_{l_j} < T < X_{m_j}, \\ m_j, & \text{if} \quad X_{m_j} < T. \end{cases}$$

Moreover, let C_j denotes the recorded lifetimes for all survived components, then C_j can also be expressed as,

$$C_j = \begin{cases} X_{l_j}, & \text{if} \quad X_{l_j} > T, \\ T, & \text{if} \quad X_{l_j} < T < X_{m_j}, \\ X_{m_j}, & \text{if} \quad X_{m_j} < T. \end{cases}$$

Next, the item's hazard failure rate under stress is determined by $H_2\left(x\right) = \lambda H_1\left(x\right)$, the acceleration factor is denoted by λ and $(\lambda > 1)$. Assume the test item is lifespan uses two parameters Kw distribution. Under accelerated condition, the HRF, SF, CDF and PDF of total lifetime x are obtained, respectively by,

$$\begin{aligned} H_2\left(x\right) &= \frac{\alpha\beta\lambda x^{\alpha-1}}{\left(1 - x^{\alpha}\right)}, & 0 < x < 1, \quad (\alpha, \beta > 0), \quad \lambda > 1, \\ S_2\left(x\right) &= \left(1 - x^{\alpha}\right)^{\lambda\beta}, & 0 < x < 1, \quad (\alpha, \beta > 0), \quad \lambda > 1, \\ F_2(x; \alpha, \beta, \lambda) &= 1 - \left(1 - x^{\alpha}\right)^{\lambda\beta}, & 0 < x < 1, \quad (\alpha, \beta > 0), \quad \lambda > 1, \\ f_2(x; \alpha, \beta, \lambda) &= \alpha\beta\lambda x^{\alpha-\mathbf{1}} \left(1 - x^{\alpha}\right)^{\beta\lambda-\mathbf{1}}, & 0 < x < 1, \quad (\alpha, \beta > 0), \quad \lambda > 1. \end{aligned} \tag{2}$$

Suppose that,

$$X = \{X_{(j1)} \leq X_{(j2)} \leq \cdots \leq X_{(jD_j)}\}, \qquad j = 1, 2,$$

be two Type-I GHCS from two categories, the $CDFs$ and $PDFs$ of which are provided by (1) and (2). Then, the likelihood function based on CSPALTs model with Type-I GHCS is then given by,

$$L(\alpha, \beta, \lambda | \underline{x}) = \prod_{j=1}^{2} \frac{n_j!}{(n_j - D_j)!} \left(1 - F_j\left(C_j\right)\right)^{n_j - D_j} \prod_{i=1}^{D_j} f\left(x_i\right). \tag{3}$$

3 ML Estimation

Our goal in this part is to obtain maximum likelihood estimates (MLEs) for α and β and the acceleration factor λ in accordance with the data $X_{(j1)} \leq X_{(j2)} \leq \cdots \leq X_{(jD_j)}$, $j = 1, 2$, obtained in presences Type-I GHCS with CSPALTs. Likelihood function may be simplified as the following expression,

$$L(\alpha,\beta,\lambda|\underline{x}) \propto (\alpha\beta)^{D_1+D_2}\,\lambda^{D_2}\prod_{j=1}^{2}\left(\left[1-x_{D_j}^{\alpha}\right]^{\delta_j\beta(n_j-D_j)}\prod_{i=1}^{D_j}x_{ji}^{\alpha-1}\left(1-x_{ji}^{\alpha}\right)^{\delta_j\beta-1}\right)$$
$$\propto (\alpha\beta)^{D_1+D_2}\,\lambda^{D_2}\left[1-x_{D_1}^{\alpha}\right]^{\beta(n_1-D_1)}\left[1-x_{D_2}^{\alpha}\right]^{\beta\lambda(n_2-D_2)}\prod_{j=1}^{2}\prod_{i=1}^{D_j}x_{ji}^{\alpha-1}\left(1-x_{ji}^{\alpha}\right)^{\delta_j\beta-1}, \tag{4}$$

where

$$\delta_j=\begin{cases}1, & \text{when} \quad j=1,\\ \lambda, & \text{when} \quad j=2.\end{cases}$$

The likelihood function's natural logarithm is supplied by,

$$\begin{aligned}\ell &= \log L(\alpha,\beta,\lambda|\underline{x})\\ &= (D_1+D_2)\log\alpha+(D_1+D_2)\log\beta+D_2\log\lambda+\beta\,(n_1-D_1)\log\left[1-x_{D_1}^{\alpha}\right]\\ &\quad+\beta\lambda\,(n_2-D_2)\log\left[1-x_{D_2}^{\alpha}\right]+\sum_{j=1}^{2}\sum_{i=1}^{D_j}(\alpha-1)\log x_{ji}+\sum_{j=1}^{2}(\delta_j\beta-1)\sum_{i=1}^{Dj}\log\left(1-x_{ji}^{\alpha}\right).\end{aligned} \tag{5}$$

With respect to α,β and acceleration factor λ, the first partial derivatives of (5) equivalent to zero to present by,

$$\begin{aligned}\frac{\partial\ell}{\partial\alpha} &= \frac{(D_1+D_2)}{\alpha}-\beta\,(n_1-D_1)\,\frac{x_{D_1}^{\alpha}\log x_{D_1}}{1-x_{D_1}^{\alpha}}-\beta\lambda\,(n_2-D_2)\,\frac{x_{D_2}^{\alpha}\log x_{D_2}}{1-x_{D_2}^{\alpha}}+\sum_{j=1}^{2}\sum_{i=1}^{D_j}\log x_{ji}\\ &\quad+\sum_{j=1}^{2}\sum_{i=1}^{D_j}(\delta_j\beta-1)\,\frac{x_{ji}^{\alpha}\log x_{ji}}{\left(1-x_{ji}^{\alpha}\right)}=0,\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\beta} &= \frac{(D_1+D_2)}{\beta}+(n_1-D_1)\log\left[1-x_{D_1}^{\alpha}\right]+\lambda\,(n_2-D_2)\log\left[1-x_{D_2}^{\alpha}\right]\\ &\quad+\sum_{j=1}^{2}\sum_{i=1}^{D_j}\delta_j\log\left(1-x_{ji}^{\alpha}\right)=0,\end{aligned} \tag{7}$$

and

$$\frac{\partial\ell}{\partial\lambda}=\frac{D_2}{\lambda}+\beta\,(n_2-D_2)\log\left[1-x_{D_2}^{\alpha}\right]+\sum_{j=1}^{2}\sum_{i=1}^{Dj}\delta_j^{*}\beta\log\left(1-x_{ji}^{\alpha}\right)=0, \tag{8}$$

where δ_j^* is the derivative of δ_j,

$$\delta_j^{*}=\begin{cases}0, & \text{when} \quad j=1,\\ 1, & \text{when} \quad j=2.\end{cases}$$

It is seen that there is no analytical solution for the non-linear (6)−(8). As a result, numerical techniques like the Newton-Raphson method are applied.

4 The Bayes Inference

The main topic in Bayesian approach is formulating the prior information which is not usually available in experimental data. The Bayes estimate for the parameters α, β and acceleration factor λ was derived in this section, assuming two different loss functions where SEL and LINEX loss functions. Supposed that an experiment based on Type-I GHCS with CSPALTs from (Kw) distribution, the parameters α, β are assumed to be independent and have gamma prior distribution as follows,

$$\pi_1(\alpha) \propto \alpha^{a_1-1}e^{-b_1\alpha}, \qquad \alpha > 0,$$

$$\pi_2(\beta) \propto \beta^{a_2-1}e^{-b_2\beta}, \qquad \beta > 0,$$

and acceleration factor λ is truncated Gamma distribution as follows,

$$\pi_3(\lambda) \propto \lambda^{a_3-1}e^{-b_3\lambda}, \qquad \lambda > 1.$$

The joint prior PDF equation of the parameters α, β and acceleration factor λ is given by,

$$\begin{aligned}\Pi(\alpha, \beta, \lambda|\underline{x}) &= \pi_1(\alpha)\pi_2(\beta)\pi_3(\lambda)\\ &\propto \alpha^{a_1-1}\beta^{a_2-1}\lambda^{a_3-1}e^{-(b_1\alpha+b_2\beta+b_3\lambda)}.\end{aligned} \tag{9}$$

The Gamma distribution is widely applied in various fields such as business, science, and engineering due to its positive and skewed distribution. Also, we used the gamma prior distributions because it is known that the family of Gamma distributions is flexible enough to cover a large variety of prior beliefs of the experimenter. So, the joint posterior PDF of unknown parameters α and β and acceleration factor λ given by,

$$\Pi^*(\alpha, \beta, \lambda|\underline{x}) = \frac{L(\alpha, \beta, \lambda|\underline{x})\Pi(\alpha, \beta, \lambda|\underline{x})}{\displaystyle\int_1^\infty\int_0^\infty\int_0^\infty L(\alpha, \beta, \lambda|\underline{x})\Pi(\alpha, \beta, \lambda|\underline{x})d\alpha d\beta d\lambda}. \tag{10}$$

By substituting from (4) and (9) in (10), we have

$$\begin{aligned}\Pi^*(\alpha, \beta, \lambda|\underline{x}) = A^{-1}&\alpha^{D_1+D_2+a_1-1}\beta^{D_1+D_2+a_2-1}\lambda^{D_2+a_3-1}\left[1 - x_{D_1}^{\alpha}\right]^{\beta(n_1-D_1)}\left[1 - x_{D_2}^{\alpha}\right]^{\beta\lambda(n_2-D_2)}\\ &\times e^{-(b_1\alpha+b_2\beta+b_3\lambda)}\prod_{i=1}^{2}\prod_{i=1}^{D_j} x_{ji}^{\alpha-1}\left(1 - x_{ji}^{\alpha}\right)^{\delta_j(\lambda)\beta-1},\end{aligned} \tag{11}$$

where A is a normalizing constant given by,

$$A = \int_1^\infty\int_0^\infty\int_0^\infty L(\alpha, \beta, \lambda|\underline{x})\Pi(\alpha, \beta, \lambda|\underline{x})d\alpha d\beta d\lambda.$$

The Bayes estimate of a function $\Omega(\alpha, \beta, \lambda)$ according to the SEL function is obtained by,

$$\begin{aligned}\hat{\Omega}_{BS} &= E(\Omega(\alpha, \beta, \lambda))\\ &= \int_1^\infty\int_0^\infty\int_0^\infty \Omega(\alpha, \beta, \lambda)\Pi^*(\alpha, \beta, \lambda|\underline{x})d\alpha d\beta d\lambda.\end{aligned} \tag{12}$$

Under LINEX loss function Bayes estimate for $\Omega(\alpha, \beta, \lambda)$ is obtained by,

$$\begin{aligned}\hat{\Omega}_{BL} &= \frac{-1}{c}\log E(e^{-c\Omega(\alpha,\beta,\lambda)})\\ &= \frac{-1}{c}\log\left[\int_1^\infty\int_0^\infty\int_0^\infty e^{-c\Omega(\alpha,\beta,\lambda)}\Pi^*(\alpha, \beta, \lambda|\underline{x})d\alpha d\beta d\lambda\right].\end{aligned} \tag{13}$$

As it turns out from (12) and (13) that parameters α, β and the acceleration factor λ estimates using the Bayesian method can not be calculated in a simple way, So for this purpose the MCMC method is used.

4.1 MCMC technique

In this part, the MCMC technique is applied for develop an empirical posterior distribution. This method may be required for creating an interval Bayes estimate considering the model's parameters and point estimate. From (11), the conditional posterior densities for α, β and acceleration factor λ is, presented as follows, respectively,

$$\begin{aligned}\Pi_1^*(\alpha|\beta,\lambda,\underline{x}) \propto\ & \alpha^{D_1+D_2+a_1-1}\left[1-x_{D_1}^{\alpha}\right]^{\beta(n_1-D_1)}\left[1-x_{D_2}^{\alpha}\right]^{\beta\lambda(n_2-D_2)} \\ & \times e^{-b_1\alpha}\prod_{i=1}^{2}\prod_{i=1}^{D_j} x_{ji}^{\alpha-1}\left(1-x_{ji}^{\alpha}\right)^{\delta_j\beta-1},\end{aligned} \tag{14}$$

$$\begin{aligned}\Pi_2^*(\beta|\alpha,\lambda,\underline{x}) \propto\ & \beta^{D_1+D_2+a_2-1}\left[1-x_{D_1}^{\alpha}\right]^{\beta(n_1-D_1)}\left[1-x_{D_2}^{\alpha}\right]^{\beta\lambda(n_2-D_2)} \\ & \times e^{-b_2\beta}\prod_{i=1}^{2}\prod_{i=1}^{D_j}\left(1-x_{ji}^{\alpha}\right)^{\delta_j\beta},\end{aligned} \tag{15}$$

and

$$\Pi_3^*(\lambda|\alpha,\beta,\underline{x}) \propto \lambda^{D_2+a_3-1}\left[1-x_{D_2}^{\alpha}\right]^{\beta\lambda(n_2-D_2)} e^{-b_3\lambda}\prod_{i=1}^{D_2}\left(1-x_{2i}^{\alpha}\right)^{\lambda\beta}. \tag{16}$$

As may be seen in (15) and (16), the conditional posterior PDF for the parameter β and acceleration factor λ have shown, that it is similar to gamma and truncated gamma density and gamma generating procedures are used to create the generation. On other hand, (14) not able to be simplified into to a standard distribution, yet its plot shows because it is somewhat typical. So, we employ the Metropolis-Hastings technique in order to generate samples for the parameter α from the conditional posterior PDF with the help of a normal proposal distribution. The following algorithms use the MH algorithm under Gibbs sampling, which is more general:

Step 1: Start with initial parameters values,

$$(\alpha^{(0)},\beta^{(0)},\lambda^{(0)}) = (\hat{\alpha}_{ML},\hat{\beta}_{ML},\hat{\lambda}_{ML}).$$

Step 2: Put $J=1$.

Step 3: Generate $\beta^{(J)}$ from Gamma,

$$\begin{aligned}& D_1+D_2+a_2\,, b_2-(n_1-D_1)\log\left[1-x_{D_1}^{\alpha}\right]-\lambda\,(n_2-D_2)\log\left[1-x_{D_2}^{\alpha}\right] \\ & -\sum_{j=1}^{2}\sum_{i=1}^{D_j}\delta_j\log\left(1-x_{ji}^{\alpha}\right).\end{aligned}$$

Step 4: In the same manner, we will produce $\lambda^{(J)}$ from truncated Gamma,

$$\left(D_2+a_{3,}\, b_3-\beta\,(n_2-D_2)\log\left[1-x_{D_2}^{\alpha}\right]-\sum_{i=1}^{D_2}\beta\log\left(1-x_{2i}^{\alpha}\right)\right)$$

Step 5: From (14), generate $\alpha^{(J)}$ with the usage of Metropolis–Hustings algorithm using typical proposal distributions $N(\alpha^{(J-1)}, \text{Var}(\hat{\alpha}))$, as follows:

i. Generate α^* from $N(\alpha^{(J-1)}, \text{Var}(\hat{\alpha}))$.

ii. Evaluate the probability of acceptance

$$p = \min\left[1, \frac{\Pi^*(\alpha^{(*)}|\beta^{(J-1)}, \lambda^{(J-1)})}{\Pi^*(\alpha^{(J-1)}|\beta^{(J-1)}, \lambda^{(J-1)})}\right]. \tag{17}$$

iii. Generate U from a uniform $(0,1)$ distribution.

iv. Accept the proposal and place it if $U \leq p$ and put $\alpha^{(J)} = \alpha^*$; else, put $\alpha^{(J)} = \alpha^{(J-1)}$.

Step 6: Put $J = J + 1$.

Step 7: **Steps** 3–6 are repeated for N times.

Step 8: The Bayes estimate under MCMC of the parameters α, β and the acceleration factor λ by using SEL function, respectively, as,

$$\begin{aligned}
\hat{\alpha}_{BS} &= \frac{1}{N-M}\sum_{i=M+1}^{N}\alpha^{(i)}, \\
\hat{\beta}_{BS} &= \frac{1}{N-M}\sum_{i=M+1}^{N}\beta^{(i)}, \\
\hat{\lambda}_{BS} &= \frac{1}{N-M}\sum_{i=M+1}^{N}\lambda^{(i)}.
\end{aligned} \tag{18}$$

where M represents how many iterations aren't considered in the (burn-in) computation. Additionally, in MCMC, the Bayes estimate for parameters α, β and acceleration factor λ by using LINEX loss function, respectively, as,

$$\begin{aligned}
\hat{\alpha}_{BL} &= \frac{-1}{c}\log[\frac{1}{N-M}\sum_{i=M+1}^{N}e^{\alpha^{(i)}}], \\
\hat{\beta}_{BL} &= \frac{-1}{c}\log[\frac{1}{N-M}\sum_{i=M+1}^{N}e^{\beta^{(i)}}], \\
\hat{\lambda}_{BL} &= \frac{-1}{c}\log[\frac{1}{N-M}\sum_{i=M+1}^{N}e^{\lambda^{(i)}}].
\end{aligned} \tag{19}$$

Step 9: We have order

$$\begin{aligned}
&\alpha_{(1)} < \alpha_{(2)} < \cdots < \alpha_{(N)}, \\
&\beta_{(1)} < \beta_{(2)} < \cdots < \beta_{(N)}, \\
&\lambda_{(1)} < \lambda_{(2)} < \cdots < \lambda_{(N)}.
\end{aligned}$$

Then the $100(1-\gamma)\%$ symmetric credible intervals of α, β and the acceleration factor λ are

$$\left(\alpha_{N(\gamma/2)}, \alpha_{N(1-\gamma/2)}\right), \quad \left(\beta_{N(\gamma/2)}, \beta_{N(1-\gamma/2)}\right), \quad \left(\lambda_{N(\gamma/2)}, \lambda_{N(1-\gamma/2)}\right). \tag{20}$$

5 Confidence intervals

5.1 Approximate CIs

The approximate Fisher information matrix $I(\alpha, \beta, \lambda)$ can be used to get the estimated CIs for parameters α, β and acceleration factor λ. according to the big sample approximation,

$$\mathbf{I}\left(\hat{\alpha}_{ML}, \hat{\beta}_{ML}\hat{\lambda}_{ML}\right) = \begin{bmatrix} -\frac{\partial^2 \ell}{\partial \alpha^2} & -\frac{\partial^2 \ell}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 \ell}{\partial \beta \partial \alpha} & -\frac{\partial^2 \ell}{\partial \beta^2} & -\frac{\partial^2 \ell}{\partial \beta \partial \lambda} \\ -\frac{\partial^2 \ell}{\partial \lambda \partial \alpha} & -\frac{\partial^2 \ell}{\partial \lambda \partial \beta} & -\frac{\partial^2 \ell}{\partial \lambda^2} \end{bmatrix}_{\hat{\alpha}_{ML}, \hat{\beta}_{ML}, \hat{\lambda}_{ML}} . \tag{21}$$

The diagonal elements of $\mathbf{I}^{-1}\left(\hat{\alpha}_{ML}, \hat{\beta}_{ML}\hat{\lambda}_{ML}\right)$ provides parametric asymptotic variances α, β and acceleration factor λ respectively. where the log-likelihood function's second partial derivatives (5), are obtained by,

$$\frac{\partial^2 \ell}{\partial \alpha^2} = \begin{cases} -\frac{(D_1 + D_2)}{\alpha^2} - \beta\,(n_1 - D_1)\left[\frac{x_{D_1}^{\alpha}\left(\log x_{D_1}\right)^2}{1 - x_{D_1}^{\alpha}} + \frac{\left(x_{D_1}^{\alpha}\log x_{D_1}\right)^2}{\left(1 - x_{D_1}^{\alpha}\right)^2}\right] \\ -\beta\lambda\,(n_2 - D_2)\left[\frac{x_{D_2}^{\alpha}\left(\log x_{D_2}\right)^2}{1 - x_{D_2}^{\alpha}} + \frac{\left(x_{D_2}^{\alpha}\log x_{D_2}\right)^2}{\left(1 - x_{D_2}^{\alpha}\right)^2}\right] \\ +\sum_{j=1}^{2}\sum_{i=1}^{D_j}(\delta_j(\lambda)\beta - 1)\left[\frac{x_{ji}^{\alpha}\left(\log x_{ji}\right)^2}{1 - x_{ji}^{\alpha}} + \frac{\left(x_{ji}^{\alpha}\log x_{ji}\right)^2}{\left(1 - x_{ji}^{\alpha}\right)^2}\right], \end{cases} \tag{22}$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \alpha} = \frac{\partial^2 \ell}{\partial \alpha \partial \beta} = \begin{cases} -\,(n_1 - D_1)\,\frac{x_{D_1}^{\alpha}\log x_{D_1}}{1 - x_{D_1}^{\alpha}} - \lambda\,(n_2 - D_2)\,\frac{x_{D_2}^{\alpha}\log x_{D_2}}{1 - x_{D_2}^{\alpha}} \\ +\sum_{j=1}^{2}\sum_{i=1}^{D_j}\delta_j(\lambda)\frac{x_{ji}^{\alpha}\log x_{ji}}{1 - x_{ji}^{\alpha}}, \end{cases} \tag{23}$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial \alpha} = \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = \begin{cases} -\beta\,(n_2 - D_2)\,\frac{x_{D_2}^{\alpha}\log x_{D_2}}{1 - x_{D_2}^{\alpha}} + \sum_{j=1}^{2}\sum_{i=1}^{D_j}\delta_j^{*}(\lambda)\beta\frac{x_{ji}^{\alpha}\log x_{ji}}{1 - x_{ji}^{\alpha}}, \end{cases} \tag{24}$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \lambda} = \frac{\partial^2 \ell}{\partial \lambda \partial \beta} = \begin{cases} (n_2 - D_2)\log\left(1 - x_{D_2}^{\alpha}\right) + \sum_{j=1}^{2}\sum_{i=1}^{D_j}\delta_j^{*}(\lambda)\log\left(1 - x_{ji}^{\alpha}\right), \end{cases} \tag{25}$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = \begin{cases} -\frac{(D_1 + D_2)}{\beta^2}, \end{cases} \tag{26}$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = \begin{cases} -\frac{D_2}{\lambda^2}. \end{cases} \tag{27}$$

Therefore,

$$(\hat{\alpha}_{ML}, \hat{\beta}_{ML}\hat{\lambda}_{ML}) \sim N\left[(\alpha,\beta,\lambda), \mathbf{I}^{-1}\left(\hat{\alpha}_{ML}, \hat{\beta}_{ML}\hat{\lambda}_{ML}\right)\right],$$

then two-sided confidence intervals for the $100(1-\gamma)\%$ normal approximation of α, β and acceleration factor λ can be defined as,

$$\widehat{\alpha} \pm Z_{\gamma/2}\sqrt{\text{Var}(\widehat{\alpha})}, \qquad \widehat{\beta} \pm Z_{\gamma/2}\sqrt{\text{Var}(\widehat{\beta})} \qquad \text{and} \qquad \widehat{\lambda} \pm Z_{\gamma/2}\sqrt{\text{Var}(\widehat{\lambda})}, \tag{28}$$

where $Z_{\gamma/2}$ is the right-tail probability $\gamma/2$ percentile of the standard normal distribution.

6 CIs for Bootstrap

The bootstrap technique is frequently applied to problems including interval estimation, bias estimates, testing hypotheses, and other statistical inference-related issues in literature. The parametric and nonparametric Bootstrap methods are used to create approximations of the model parameters confidence intervals in this section, for more detail presented by Davison and Hinkley [7] and Efron and Tibshirani [9]. The Boot-p confidence interval is obtained based on the idea presented in Efron [8] and the Boot-t confidence interval is obtained based on the idea of Hall [14]. Using the Bootstrap approach, CLs can be calculated using the following algorithms:

6.1 Boot-p method

Step 1: Generate two samples Type-I GHCS $\{x_{j_1}, x_{j_2}, \dots, x_{jn_j}\}$ and $j = 1, 2$ from CDFs $F_1(x)$ and $F_2(x)$ given in (1) and (2) respectively, calculate bootstrap estimate of α, β and the acceleration factor λ symbolled by $\hat{\alpha}^*$, $\hat{\beta}^*$ and $\hat{\lambda}^*$ using (6)−(8).

Step 2: Step 1 is repeated for M^* times, we obtain

$$\left(\hat{\alpha}^{*(1)}, \hat{\alpha}^{*(2)}, \dots, \hat{\alpha}^{*(M^*)}\right), \quad \left(\hat{\beta}^{*(1)}, \hat{\beta}^{*(2)}, \dots, \hat{\beta}^{*(M^*)}\right), \quad \text{and} \quad \left(\hat{\lambda}^{*(1)}, \hat{\lambda}^{*(2)}, \dots, \hat{\lambda}^{*(M^*)}\right).$$

Step 3: With ranged to $I = 1, 2, \dots, M^*$, sort $\hat{\alpha}^{*(I)}$, $\hat{\beta}^{*(I)}$ and $\hat{\lambda}^{*(I)}$, in ascending order and obtain

$$\left(\hat{\alpha}^*_{(1)}, \hat{\alpha}^*_{(2)}, \dots, \hat{\alpha}^*_{(M^*)}\right), \quad \left(\hat{\beta}^*_{(1)}, \quad \hat{\beta}^*_{(2)}, \dots, \hat{\beta}^*_{(M^*)}\right) \quad \text{and} \quad \left(\hat{\lambda}^*_{(1)}, \quad \hat{\lambda}^*_{(2)}, \dots, \hat{\lambda}^*_{(M^*)}\right).$$

The $100(1-\gamma)\%$ percentile bootstrap-p estimate CIs for parameters α, β and the acceleration factor λ, can be defined as,

$$\begin{cases} \left(\hat{\alpha}^*_{M^*(\gamma/2)}, \hat{\alpha}^*_{M^*(1-\gamma/2)}\right), \\ \left(\hat{\beta}^*_{M^*(\gamma/2)}, \hat{\beta}^*_{M^*(1-\gamma/2)}\right), \\ \left(\hat{\lambda}^*_{M^*(\gamma/2)}, \hat{\lambda}^*_{M^*(1-\gamma/2)}\right). \end{cases}$$

6.2 Boot-t method

Step 1: Similarly as in Boot-p, we will generate two samples G-1 HCS $\{x_{j_1}, x_{j_2}, \dots, x_{jn_j}\}$ and $j = 1, 2$ and get,

$$\left(\hat{\alpha}^{*(1)}, \hat{\alpha}^{*(2)}, \dots, \hat{\alpha}^{*(M^*)}\right), \quad \left(\hat{\beta}^{*(1)}, \hat{\beta}^{*(2)}, \dots, \hat{\beta}^{*(M^*)}\right) \quad \text{and} \quad \left(\hat{\lambda}^{*(1)}, \hat{\lambda}^{*(2)}, \dots, \hat{\lambda}^{*(M^*)}\right).$$

Step 2: We will find the order statistics $\left(\hat{\Theta}^*_{(1)}, \hat{\alpha}^*_{(2)}, \ldots, \hat{\Theta}^*_{(M^*)}\right)$, where $\Theta = (\alpha, \beta, \lambda)$,

$$T^*_{L(h)} = \frac{\left(\hat{\Theta}^*_{(h)} - \hat{\Theta}\right)}{\sqrt{\text{Var}(\hat{\Theta}^*_{(h)})}}, \qquad h = 1, 2, \ldots, M^*, \qquad L = 1, 2, 3, \qquad \Theta = (\alpha, \beta, \lambda).$$

Step 3: Let $G_L(x) = P(T^*_L \leq x)$ be the CDF of T^*_L. For a given x, define as,

$$\hat{\Theta}_{Bt}(x) = \hat{\Theta} + \sqrt{\text{Var}(\hat{\Theta})} G_L^{-1}(x), \qquad L = 1, 2, 3.$$

The estimated CIs for the $100(1-\gamma)\%$ percentile bootstrap-t of Θ can be defined as,

$$\left(\hat{\Theta}_{Bt}(\gamma/2), \hat{\Theta}_{Bt}(1-\gamma/2)\right).$$

7 Illustrative Example

To demonstrate the analytical methods that were created in the prior parts, we produce two samples Type-I GHCS from Kumaraswamy (Kw) distribution, one of them based on usual usage and the other based on an accelerated usage, using $(\alpha, \beta, \lambda) = (2.1, 1.7, 1.5)$, $l_1 = 30$, $l_2 = 35$, $m_1 = 40$, $m_2 = 45$, $n_1 = n_2 = 50$ and $T = 0.85$. The following two sample sets were observed:

$$\begin{aligned}
\text{Set 1 "normal"} : \{&0.165934, 0.170091, 0.21523, 0.251006, 0.262728, 0.264426, 0.287539, 0.304589,\\
&0.342255, 0.395256, 0.398959, 0.402432, 0.409258, 0.44814, 0.469691, 0.472408,\\
&0.513187, 0.515024, 0.515875, 0.546612, 0.554279, 0.560665, 0.562383, 0.562864,\\
&0.588835, 0.591895, 0.594287, 0.611738, 0.63302, 0.646672, 0.653424, 0.664639,\\
&0.671507, 0.753876, 0.772386, 0.808161, 0.812272, 0.815781, 0.824898, 0.831484,\\
&0.835482, 0.863766, 0.868872, 0.910629, 0.915238, 0.917406, 0.918123, 0.923585,\\
&0.923812, 0.942806\}.
\end{aligned}$$

$$\begin{aligned}
\text{Set 2 "accelerated"} : \{&0.0373403, 0.0561753, 0.077267, 0.0898709, 0.124751, 0.199974, 0.222979,\\
&0.224905, 0.226023, 0.228831, 0.229171, 0.243686, 0.269752, 0.274268,\\
&0.276683, 0.329285, 0.348028, 0.358576, 0.36825, 0.397556, 0.428404,\\
&0.451983, 0.459549, 0.482801, 0.511171, 0.532534, 0.534415, 0.549035,\\
&0.567226, 0.577922, 0.582154, 0.584633, 0.612704, 0.624665, 0.64243,\\
&0.648763, 0.653921, 0.670495, 0.676667, 0.677367, 0.715224, 0.717966,\\
&0.723356, 0.752268, 0.752663, 0.773214, 0.812827, 0.822813, 0.868053,\\
&0.909937\}.
\end{aligned}$$

The MLE of α, β and the acceleration factor λ are calculated to be $\hat{\alpha}_{ML} = 2.07226$, $\hat{\beta}_{ML} = 1.80977$ and $\hat{\lambda}_{ML} = 1.18265$, respectively.

Table 1 The 95% CIs are displayed. for parameters α, β and acceleration factor λ. For calculating credible intervals for Bayesian approach based on Gibbs in presences MH algorithm to produce a MCMC with $12,000$ views. The first 2000 MCMC samples were discarded as burn-in and were not used in the subsequent analysis. Applying the Gamma priors for α, β and acceleration factor λ, since hyper parameters,

$$(a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 2).$$

Table 1: 95% estimate intervals of αand $\beta$and the acceleration factor λ.

Parameter	MCMC	ACI	Boot-p	Boot-t
α	$(1.317970, 2.164580)$	$(1.578330, 2.566190)$	$(1.657430, 2.852680)$	$(1.593830, 2.670840)$
β	$(0.884485, 1.944270)$	$(1.062750, 2.556800)$	$(1.317980, 2.893500)$	$(1.120550, 2.441260)$
λ	$(0.765431, 1.733170)$	$(0.678934, 1.686370)$	$(0.763133, 1.820230)$	$(0.717908, 1.608460)$

Table 2 shows the posterior mean, skewness (Ske), standard deviation $(S.D.)$, and median of the parameters' MCMC findings. Additionally, the Bayes MCMC estimates with respect to the $SEL$and $LINEXL$ functions with varying parameter (c) values of LINEX loss function are displayed in Table 3.

Table 2: The Bayes MCMC results.

Parameter	Mean	Median	SD	SKe
α	1.71971	1.71434	0.219246	0.261991
β	1.34608	1.32252	0.271595	0.540458
λ	1.18531	1.16061	0.271595	0.605055

Table 3: MLE and Bayes MCMC estimates under SEL and LINEX.

Parameter	MLE	SEL	LINEXL		
			$c = -3$	$c = 0.0003$	$c = 3$
α	2.07226	1.7197	1.7442	1.7197	1.6961
β	1.80977	1.3461	1.3849	1.3461	1.3109
λ	1.18265	1.1853	1.2183	1.1853	1.1555

Figure 1 show Histogram of α, β and acceleration factor λ respectively generated by MCMC method (see Figure 2). In addition, explain the convergence that occurs while using the MCMC approach to generate a posterior distribution.

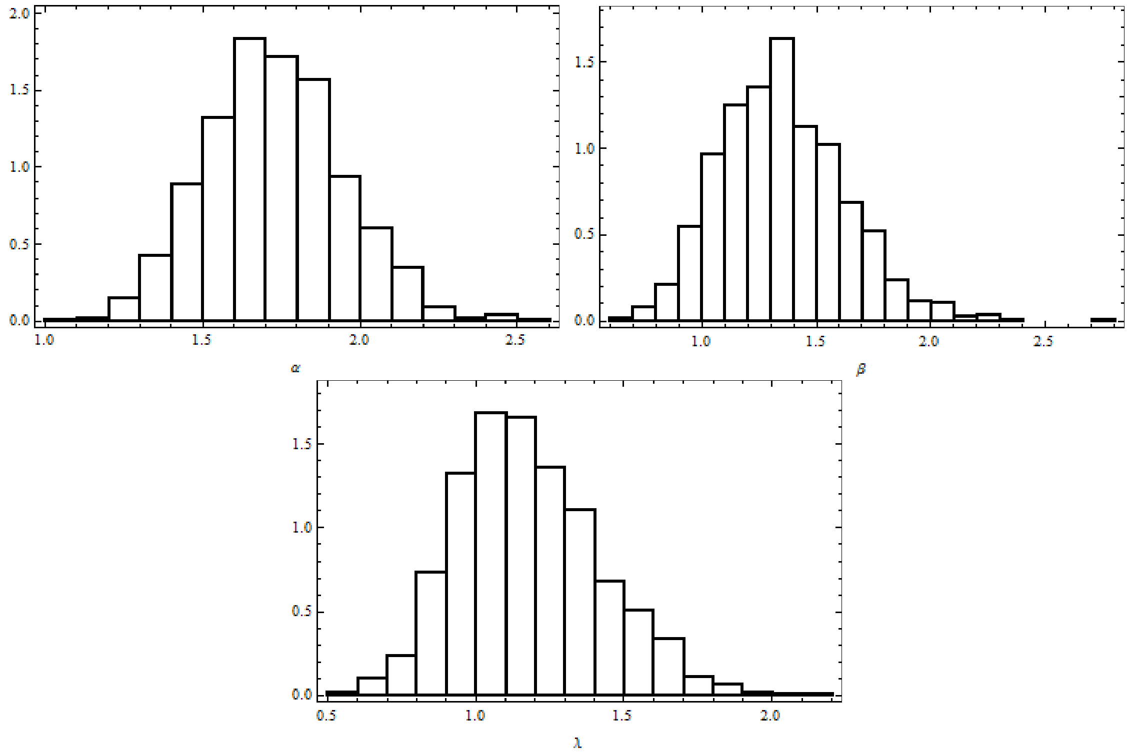

Figure 1: Histogram of α, β and acceleration factor λ respectively produced using the MCMC technique.

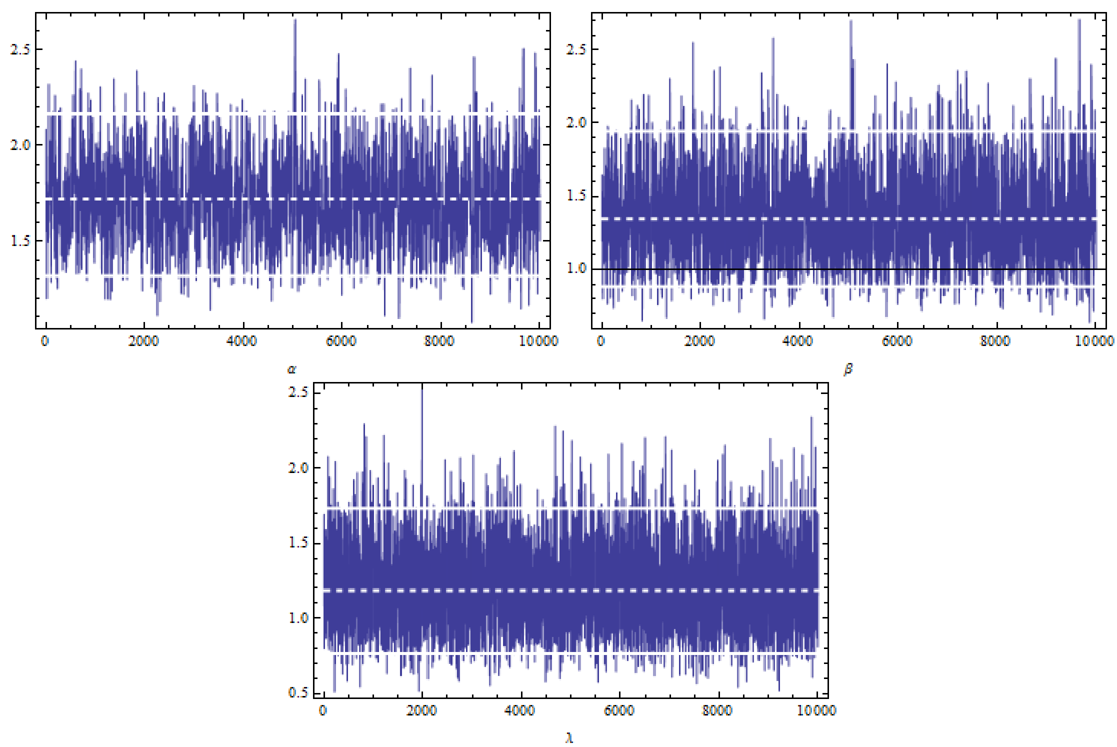

Figure 2: MCMC output of α and β, and the acceleration factor λ, respectively. Dashed lines $(\ldots)$ represent the posterior means and soled lines $(-)$ represent lower, and upper bounds 95% probability interval generated by MCMC method.

8 Simulation Study and Comparisons

This section uses a Monte Carlo simulation study to compare and assess the different estimate methods. Examined the characteristics and effectiveness of the various estimators as well. During the simulation, we employed many options for n_1, n_2, m_1, m_2 and T. As a result, in order to compare various estimating techniques, for point estimates, we looked at the average mean (AM) and mean square error (MSE), for interval estimates, we looked at the average width (AW) and coverage probability (CP). We use the Metropolis–Hastings technique of α, β and acceleration factor λ to construct a Markov chain with $12,000$ produced data points, and since the initial values have a significant impact on the first 2000 sets, we won't be taking them into consideration. $(MATHEMATICA ver.8.0)$ was used for all calculations.

The previous presumptions that α, β and acceleration factor λ follow Gamma (a_1, b_1), Gamma (a_2, b_2) and truncated Gamma (a_3, b_3) distributions, respectively. Using the parameters, we produced 1000 two samples of Type-I GHCS with sizes n_1 and n_2 from the Kumaraswamy distribution. $(\alpha, \beta, \lambda) = (2.1, 1.7, 1.5)$, different values for T, l_j, m_j where $j = \{1, 2\}$. Tables 4, 5 and 6 show the corresponding MSEs for various censoring methods as well as the estimated average mean of the MLE, Bootstrap, and Bayes estimates in relation to the SEL and LINEX loss functions of the parameters α, β and acceleration factor λ, and Tables 7, 8 and 9 the mean length and the associated coverage probability, respectively.

Table 4: Mean (first row) and MSE (second row) α, β, and λ, with $c = 0.0001, 1, -1$.

(n_1,n_2)	T	P	MLE	MCMC				Bootstrap	
(m_1,m_2)				SEL	LINEXL			Boot-p	Boot-t
(l_1,l_2)					$c=-1$	$c=0.0001$	$c=1$		
(35,25)		α	2.251800	2.173900	2.177500	2.173800	2.170200	2.425800	2.359800
(30,20)	0.15		0.197671	0.175742	0.176988	0.175730	0.174536	0.305649	0.259015
(25,15)		β	2.005700	1.847000	2.011200	1.845600	1.721000	2.397400	2.088200
			0.563025	0.320099	0.538085	0.318545	0.202786	1.286670	0.665109
		λ	1.664300	1.486000	1.530800	1.485600	1.445500	1.907100	1.743100
			0.413037	0.296002	0.316009	0.295856	0.285611	0.801232	0.490244
(45,45)		α	2.208100	2.143000	2.209300	2.142900	2.140600	2.328400	2.284200
(35,40)	0.15		0.127385	0.119539	0.121826	0.119533	0.118986	0.180071	0.158261
(25,30)		β	1.946500	2.069400	2.145300	2.068200	1.958100	2.275600	2.004800
			0.479859	0.463564	0.469102	0.461693	0.390250	0.996844	0.542941
		λ	1.589500	1.326300	1.360200	1.326000	1.295200	1.673400	1.633300
			0.333120	0.320030	0.329421	0.320061	0.315164	0.336928	0.278975
(50,50)		α	2.201500	2.148600	2.163100	2.148500	2.146600	2.299200	2.262900
(40,45)	0.15		0.090645	0.080247	0.089578	0.080243	0.079862	0.125950	0.110906
(30,35)		β	1.921000	2.045200	2.150600	2.044100	1.948700	2.166400	1.965900
			0.374627	0.366234	0.370643	0.366751	0.352204	0.677799	0.407244
		λ	1.541800	1.321900	1.350000	1.321600	1.295800	1.609200	1.576700
			0.160257	0.158483	0.159916	0.158537	0.151304	0.199576	0.172381

Table 5: Mean (first row) and MSE (second row) α, β, and λ, with $c = 0.0001, 1, -1$.

(n_1, n_2)	T	P	MLE	MCMC				Bootstrap	
(m_1, m_2)				SEL	LINEXL			Boot-p	Boot-t
(l_1, l_2)					$c=-1$	$c=0.0001$	$c=1$		
(35,25)		α	2.253500	2.175800	2.179400	2.175800	2.172300	2.427600	2.361300
(30,20)	0.2		0.183896	0.166806	0.167841	0.166796	0.165789	0.290475	0.2441100
(25,15)		β	1.965200	1.874200	2.044100	1.872700	1.744400	2.341500	2.044600
			0.483421	0.371943	0.620171	0.370173	0.237643	1.099740	0.569626
		λ	1.709400	1.458200	1.500800	1.457800	1.419400	1.969600	1.790800
			0.543532	0.286427	0.302814	0.286316	0.279227	1.051860	0.649288
(45,45)		α	2.223600	2.156400	2.198400	2.156300	2.154000	2.341700	2.297600
(35,40)	0.2		0.135694	0.124979	0.742966	0.124974	0.124506	0.189541	0.166727
(25,30)		β	1.943000	2.060400	2.158700	2.059200	1.950300	2.265200	1.998400
			0.454662	0.415959	0.425476	0.415974	0.369608	0.847736	0.452141
		λ	1.576700	1.324800	1.357900	1.324500	1.294400	1.656200	1.619000
			0.310774	0.295579	0.289758	0.295655	0.274505	0.272000	0.231717
(50,50)		α	2.185300	2.122400	2.185500	2.122400	2.120500	2.285300	2.249100
(40,45)	0.2		0.109411	0.108943	0.112425	0.108941	0.108788	0.144085	0.129668
(30,35)		β	1.87700	2.065600	2.124300	2.064600	1.967600	2.120000	1.925300
			0.340004	0.332464	0.339112	0.330927	0.323745	0.614569	0.379149
		λ	1.555800	1.282400	1.308600	1.282100	1.257800	1.622500	1.593800
			0.1562059	0.157157	0.169680	0.157242	0.150516	0.201204	0.176174

Table 6: Mean (first row) and MSE (second row) α, β, and λ, with $c = 0.0001, 1, -1$.

(n_1, n_2)	T	P	MLE	MCMC				Bootstrap	
(m_1, m_2)				SEL	LINEXL			Boot-p	Boot-t
$(l_1,\ l_2)$					$c=-1$	$c=0.0001$	$c=1$		
(35,25)		α	2.256300	2.184200	2.187600	2.184200	2.180900	2.429900	2.363800
(30,20)	0.3		0.185683	0.179752	0.180556	0.179744	178984	0.291108	0.245145
(25,15)		β	1.957800	1.876400	2.046200	1.874900	1.746600	2.337700	2.038900
			0.540166	0.367979	0.611682	0.366238	0.235740	1.220290	0.636698
		λ	1.700900	1.460000	1.503400	1.459600	1.420700	1.949900	1.782400
			0.451587	0.337345	0.364288	0.337146	0.322682	0.854456	0.539159
(45,45)		α	2.186700	2.112500	2.184200	2.112500	2.110100	2.305500	2.261900
(35,40)	0.3		0.121410	0.107994	0.691574	0.107990	0.107551	0.166795	0.147578
(25,30)		β	1.915100	2.048100	2.114900	2.046900	1.939500	2.233900	1.9710000
			0.480312	0.477900	0.108455	0.476219	0.340065	0.897397	0.491783
		λ	1.563600	1.296100	1.328400	1.295800	1.266300	1.644500	1.606300
			0.181678	0.178192	0.183178	0.178262	0.176627	0.238296	0.199746
(50,50)		α	2.208200	2.146700	2.192300	2.146700	2.144600	2.308900	2.272500
(40,45)	0.3		0.100415	0.094327	0.698747	0.094323	0.094001	0.139630	0.123790
(30,35)		β	1.917700	2.071400	2.148800	2.070300	1.972700	2.165300	1.966200
			0.343886	0.334220	0.354667	0.332648	0.322722	0.637656	0.386928
		λ	1.566300	1.312800	1.340200	1.312600	1.287200	1.635100	1.603800
			0.141508	0.130411	0.142822	0.130498	0.129893	0.216000	0.188352

Table 7: 95% coverage probabilities (first row) and mean lengh (second row) for α, β, and λ.

(n_1, n_2) (m_1, m_2) (l_1, l_2)	T	P	MLE	MCMC	Bootstrap Boot-p	Bootstrap Boot-t
(35,25) (30,20) (25,15)	0.15	α	0.9420 1.5246	0.2380 0.2723	0.8820 1.7221	0.8820 1.5278
		β	0.9640 2.4106	0.6240 2.0269	0.8820 3.4681	0.8820 2.2557
		λ	1.0000 2.2563	0.9640 1.0754	0.9440 3.1022	0.9440 2.1158
(45,45) (35,40) (25,30)	0.15	α	0.9400 1.2624	0.2860 0.2161	0.8960 1.3861	0.8960 1.2722
		β	0.9560 2.3053	0.5260 1.9002	0.9080 3.1646	0.9080 2.1881
		λ	0.9940 1.7180	0.8040 0.9419	0.9620 1.9734	0.9620 1.6345
(50,50) (40,45) (30,35)	0.15	α	0.9540 1.1635	0.2660 0.2003	0.9440 1.2548	0.9440 1.1712
		β	0.9700 2.0278	0.5160 1.7654	0.9120 2.5771	0.9120 1.9477
		λ	0.9760 1.5268	0.7860 0.8696	0.9760 1.7274	0.9760 1.4673

Table 8: 95% coverage probabilities (first row) and mean lengh (second row) for α, β, and λ.

(n_1, n_2) (m_1, m_2) (l_1, l_2)	T	P	MLE	MCMC	Bootstrap Boot-p	Bootstrap Boot-t
(35,25) (30,20) (25,15)	0.2	α	0.9520 1.5293	0.2720 0.2688	0.8920 1.7298	0.8920 1.5360
		β	0.9740 2.3494	0.6140 2.053	0.9020 3.3592	0.9020 2.2041
		λ	1.0000 2.3324	0.9480 1.0536	0.9300 3.2689	0.9300 2.1928
(45,45) (35,40) (25,30)	0.2	α	0.9520 1.2715	0.2620 0.2208	0.8820 1.3882	0.8820 1.2771
		β	0.9780 2.2948	0.5140 1.8910	0.9120 3.1355	0.9120 2.1873
		λ	0.9980 1.7015	0.8120 0.9393	0.9660 1.9486	0.9660 1.6219
(50,50) (40,45) (30,35)	0.2	α	0.9280 1.1557	0.2340 0.1999	0.9080 1.2445	0.9080 1.1571
		β	0.9420 1.9639	0.7940 1.7821	0.9200 2.5220	0.9200 1.8928
		λ	0.9840 1.5371	0.4560 0.8423	0.9680 1.7259	0.9680 1.4831

Table 9: 95% coverage probabilities (first row) and mean lengh (second row) for α, β, and λ.

(n_1, n_2)	T	P	MLE	MCMC	Bootstrap	
(m_1, m_2) (l_1, l_2)					Boot-p	Boot-t
(35,25)		α	0.9520	0.2460	0.8840	0.8840
(30,20)	0.3		1.5303	0.2642	1.7187	1.5249
(25,15)		β	0.9600	0.5820	0.9000	0.9000
			2.3493	2.0546	3.3881	2.2030
		λ	1.0000	0.9520	0.9400	0.9400
			2.3193	1.0510	3.1801	2.1849
(45,45)		α	0.9520	0.2860	0.8980	0.8980
(35,40)	0.3		1.2546	0.2215	1.3705	1.2579
(25,30)		β	0.9580	0.4680	0.8780	0.8780
			2.2672	1.8831	3.1004	2.1587
		λ	0.9980	0.8220	0.9780	0.9780
			1.6874	0.9216	1.9359	1.6097
(50,50)		α	0.9640	0.2480	0.8860	0.8860
(40,45)	0.3		1.1658	0.2061	1.2548	1.1681
(30,35)		β	0.9720	0.4640	0.8880	0.8880
			2.0196	1.7872	2.5797	1.9332
		λ	0.9840	0.7760	0.9700	0.9700
			1.5508	0.8646	1.7539	1.4881

9 Conclusions

In this paper, we examined the estimate of the acceleration factor and the two-parameter Kumaraswamy distribution using Type-I GHCS, under CSPALTs. Several techniques are proposed in this study to estimate the parameters of the Kumaraswamy distribution and the acceleration factor that are of interest. Using the observed FIM, ML estimates are obtained and the corresponding ACI sare determined for classical estimation. In addition, two parametric bootstrap models for point and interval estimates (Boot-p and Boot-t) are provided for comparison. For the Bayesian estimation, point and interval estimates are constructed using the MCMC technique. The SEL and LINEX loss functions are used to derive the Bayes estimates. Monte Carlo simulation studies and a numerical example are examined to demonstrate the use of the suggested point and interval estimators. It is found that given the parameters α, β and acceleration factor λ, the Bayes estimates outperform MLEs.

From the results, we took notice of the subsequent:

- For the majority of situations of n_j, $l_j$and m_j where $j = \{1, 2\}$, we can see from Tables 4–6 that the Bayes estimate in relation to SEL and LINEX loss functions based on the MCMC technique displays lower MSEs than the MLE method and Bootstrap method.
- All estimates have lower MSEs as the effective sample sizes, m_j and n_j are increased.
- In Bayes estimate under LINEX loss function, LINEX loss function with $c = 1$ is the best mode for α, β and acceleration factor λ, all based on the smallest MSEs.

- For the value of T we selected, the numerical findings are more acceptable.

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Conflicts of Interest The authors declare no conflict of interest.

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